Order Relations between Geometric Means and Power Means of Positive Definite Matrices

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Abstract

We consider different versions of geometric means of positive definite matrices and compare them with power means.

1 Introduction

The object of this note is to study the limits of some order relations between geometric means and power means of two positive definite matrices.

Let A and B be two $n \times n$ positive definite matrices. Among different notions of a geometric mean of A, B are the entities

$$F(A,B) = \left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{1}$$

$$G(A,B) = A\#B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}, \tag{2}$$

$$S(A,B) = A\natural B = (A^{-1}\#B)^{\frac{1}{2}} A\, (A^{-1}\#B)^{\frac{1}{2}}. \tag{3}$$

When A and B commute, each of these reduces to $(AB)^{1/2}$. The quantity $G(A,B)$ is the most common notion of the geometric mean with pleasing theoretical properties and myriad applications. The formula (2) was introduced by Pusz and Woronowicz [14], and its theory expanded by Kubo and Ando [13]. We call $G(A,B)$ *the geometric mean* of A and B. The mean $S(A,B)$ was introduced by Fiedler and Pták [10]. It is called the *spectral geometric mean*, and one of its important properties is that its eigenvalues are the same as those of $(AB)^{1/2}$. (Here the square root $(AB)^{1/2}$ is the unique square root of AB with positive eigenvalues.) This spectral property is also shared by $F(A,B)$. The quantity $\operatorname{tr} F(A,B)$ is called the *fidelity* between A and B and is important in quantum information. We call $F(A,B)$ the *operator fidelity.*

All the three quantities F, G, S are continuous in A and B. Hence their definitions can be extended to positive semidefinite matrices A, B. While G and S are symmetric in A and B, F is not. However, $F(A,B)$ and $F(B,A)$ have the same eigenvalues. We express this by writing

$$F(A,B) =_\lambda F(B,A). \tag{4}$$

We have remarked above that

$$S(A,B) =_\lambda F(A,B) =_\lambda (AB)^{1/2}. \tag{5}$$

For any real number p, we define the *p-power mean* of A and B as

$$Q_p(A,B) = \left(\frac{A^p + B^p}{2}\right)^{1/p}, \tag{6}$$

with the understanding that

$$Q_0(A,B) = \lim_{p\to 0} Q_p(A,B).$$

It is known [2] that

$$Q_0(A,B) = \exp\left(\frac{\log A + \log B}{2}\right), \tag{7}$$

and this is called the *log Euclidean mean* of A and B, another version of the geometric mean. Note that $Q_1(A,B)$ is just the arithmetic mean $\frac{A+B}{2}$.

When a and b are positive numbers, $Q_p(a,b)$ is an increasing function of p on $\mathbb{R}$. In particular,

$$\sqrt{a\,b} \le \left(\frac{a^p+b^p}{2}\right)^{1/p}, \quad p \ge 0. \tag{8}$$

For matrices the situation is much more intricate, with several notions of geometric mean and of order.

We will consider three different order relations on the cone $\mathbb{P}$ of positive definite matrices. The usual *Loewner order* is defined by saying $A \le B$ if $B-A$ is positive semidefinite (i.e., if $B-A \ge 0$). Let $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$ be the decreasingly ordered eigenvalues of A. Then $A \le B$ implies $\lambda_j(A) \le \lambda_j(B)$ for all $1 \le j \le n$. We will say that

$$A \le_\lambda B \quad \text{if} \quad \lambda_j(A) \le \lambda_j(B), \quad 1 \le j \le n, \tag{9}$$

and call this the *spectral order* on $\mathbb{P}$. (cf. the notation (5) introduced earlier.)

Another order that has been introduced by Dumitru and Franco [9] recently is the *near order* defined as

$$A \preceq B \quad \text{if} \quad A\#B^{-1} \le I. \tag{10}$$

Among its several interesting properties is that it sits between the Loewner order and the spectral order:

$$A \le B \Longrightarrow A \preceq B \Longrightarrow A \le_\lambda B. \tag{11}$$

See [9], [11]. The order $\preceq$ is not transitive. However, the following implication has been shown in [9]:

$$A \preceq B \text{ and } B \le C \Longrightarrow A \preceq C. \tag{12}$$

For fixed A and B consider $Q_p(A,B)$ as a matrix function of $p \in \mathbb{R}$. This has interesting order properties with respect to the three orders introduced above. It was shown by Bhagwat and Subramanian [2] that with respect to the Loewner order $Q_p(A,B)$ is a monotone function of p on $(-\infty,-1]$ and $[1,\infty)$ but not on $(-1,1)$. Bhatia and Grover [4] observed that, in contrast, with respect to the spectral order, $Q_p(A,B)$ is a monotone function of p on the whole real line $\mathbb{R}$. In other words

$$p < p' \Longrightarrow Q_p(A,B) \le_\lambda Q_{p'}(A,B). \tag{13}$$

In a recent paper, Hwang and Kim [12] have proved the stronger result that this last statement is true also when the spectral order $\leq_\lambda$ is replaced by the near order $\prec$.

Now we can state the main results of this note. In [5] Bhatia and Kittaneh established a matrix version of the arithmetic-geometric mean inequality (AGM). Let $s_j(A), 1 \leq j \leq n$ be the decreasingly ordered singular values of the matrix A. The inequality of Bhatia and Kittaneh says that for all positive semidefinite matrices A and B we have

$$s_j(A^{1/2}B^{1/2}) \leq s_j\left(\frac{A+B}{2}\right). \tag{14}$$

In their subsequent works [6], [7] these authors extended this in several directions and conjectured another stronger version of the AGM; viz.

$$s_j^{1/2}(AB) \leq s_j\left(\frac{A+B}{2}\right). \tag{15}$$

This was proved by Drury [8]. For our present discussion it is convenient to formulate these inequalities in another way. It is easy to see that inequalities (14) and (15) are equivalent to the statements

$$F(A,B) \leq_\lambda Q_1(A,B), \tag{16}$$

and

$$F(A,B) \leq_\lambda Q_{1/2}(A,B), \tag{17}$$

respectively. The monotonicity relation (13) shows that $F(A,B) \leq_\lambda Q_p(A,B)$ for all $p \geq 1/2$, and raises the natural question whether this might be the case for any $0 < p < 1/2$. Our first proposition answers this.

Proposition 1. *The inequality*

$$F(A,B) \leq_\lambda Q_p(A,B), \tag{18}$$

is valid for all positive semidefinite matrices A and B if and only if $p \geq 1/2$.

Our next question is whether the AGM (16) or (17) could be strengthened in another direction: can we replace the spectral order $\leq_\lambda$ by the near order $\prec$? In view of Proposition 1 this cannot be so if $0 < p < 1/2$. Our next proposition says this cannot be the case for any $p > 0$.

Proposition 2. *For any $p>0$ there exist positive definite matrices A and B for which the inequality $F(A,B) \preceq Q_p(A,B)$ is not true.*

The import of Propositions 1 and 2 is that the inequality (17) cannot be improved either by replacing the parameter $1/2$ by a smaller value of p or by replacing the order $\leq_\lambda$ by a stronger order $\preceq$.

Similar questions could be asked for the spectral geometric mean. Since $S(A,B) =_\lambda F(A,B)$, Proposition 1 shows $S(A,B) \leq_\lambda Q_p(A,B)$ for all A and B if $p \geq 1/2$ and this is not so if $p < 1/2$. Gan and Huang [11] have shown that we do have

$$S(A,B) \preceq Q_1(A,B), \tag{19}$$

for all positive definite A, B. We know that with respect to the order $\leq$ $Q_p(A,B)$ is an increasing function of p on $[1,\infty)$. So from (12) and (19) it follows that $S(A,B) \preceq Q_p(A,B)$ for all $p \geq 1$. This leaves the question whether $S(A,B) \preceq Q_p(A,B)$ for $1/2 \leq p < 1$. In other words, do we have the same kind of strengthening of (19) as the inequality (17) is for (16)? Our next proposition says that the answer is no.

Proposition 3. *For each $1/2 \leq p < 1$ there exist positive definite matrices A and B for which the inequality $S(A,B) \preceq Q_p(A,B)$ is not true.*

Finally, we ask a similar question about the geometric mean $G(A,B) = A\#B$. Among several attractive features of this mean is that it satisfies the strong AGM: we have $G(A,B) \leq Q_1(A,B)$ in the Loewner order. On the other hand Bhatia and Grover [4] have observed that the inequality $G(A,B) \leq_\lambda Q_0(A,B)$ is not always true. Might this be true if Q_0 is replaced by Q_p for any $0 < p < 1$? The answer is no.

Proposition 4. *For each $0 \leq p < 1$ there exist positive definite matrices A and B for which the inequality $G(A,B) \leq_\lambda Q_p(A,B)$ is not true.*

2 Proofs

Each of our propositions is proved by constructing an appropriate example.

Example 1. Choose and fix $0 < p < 1/2$. For $\alpha > 1$ consider the unit vectors

$$u = \frac{1}{\sqrt{1+\alpha^2}} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \quad \text{and} \quad v = \frac{1}{\sqrt{1+\alpha^2}} \begin{bmatrix} \alpha \\ 1 \end{bmatrix}.$$

Let

$$\gamma = u^* v = \frac{2\alpha}{1+\alpha^2}.$$

Note that the map $\alpha \longmapsto \gamma$ is a bijection from $(1, \infty)$ onto $(0, 1)$. Let A and B be the matrices defined as

$$A = uu^* = \frac{1}{1+\alpha^2} \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{bmatrix}, \quad B = vv^* = \frac{1}{1+\alpha^2} \begin{bmatrix} \alpha^2 & \alpha \\ \alpha & 1 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix}.$$

Since A and B are orthogonal projections, we have $A^p = A$ and $B^p = B$ for all $p > 0$. So

$$F(A, B) = (A^{1/2} B A^{1/2})^{1/2} = (uu^* vv^* uu^*)^{1/2} = (\gamma^2 uu^*)^{1/2} = \gamma A.$$

Hence,

$$\lambda_1^p(F(A, B)) = \gamma^p.$$

On the other hand,

$$\lambda_1^p(Q_p(A, B)) = \lambda_1\left(\frac{A^p + B^p}{2}\right) = \lambda_1\left(\frac{A+B}{2}\right) = \frac{1}{2}\lambda_1(A+B) = \frac{1+\gamma}{2}.$$

Let $f(\gamma) = \gamma^p - \frac{1}{2}(1+\gamma)$. If we show that there exists a γ in $(0, 1)$ for which $f(\gamma) > 0$, then it would follow that there exist matrices A and B for which

$$\lambda_1(F(A, B)) > \lambda_1(Q_p(A, B)),$$

and therefore for these matrices we do not have the inequality $F(A, B) \leq_\lambda Q_p(A, B)$. This is a simple calculus problem. Note that $f(1) = 0$ and $f'(1) = p - 1/2 < 0$ (since $0 < p < 1/2$). So there exist $\gamma < 1$ for which $f(\gamma) > 0$. This proves Proposition 1.

Now we turn to Proposition 4, our other statement that involves the spectral order. Some comments are in order before we present an example.

By a theorem of Ando and Hiai [1] we have the log majorisation $\lambda(G(A,B)) \prec_{w\log} \lambda(Q_0(A,B))$. So, in particular,

$$\lambda_1(G(A,B)) \le \lambda_1(Q_0(A,B)) \le \lambda_1(Q_p(A,B))$$

for all $p \ge 0$. So, to establish Proposition 4 we will have to look at eigenvalues λ_j for $j > 1$. We will show that for every $0 < p < 1$, there exist 2×2 positive definite matrices A and B for which

$$\lambda_2(G(A,B)) > \lambda_2(Q_p(A,B)).$$

We will need the following simple lemmas.

Lemma 1. *Let A be a positive semidefinite matrix of rank 1 and B any positive definite matrix. Then*

$$G(A,B) = \frac{A}{\sqrt{\operatorname{tr} AB^{-1}}}.$$

Proof. The geometric mean $G(A,B)$ can also be expressed as

$$G(A,B) = (AB^{-1})^{1/2} B.$$

See [3] p. 110, where $(AB^{-1})^{1/2}$ is the unique square root of AB^{-1} with nonnegative eigenvalues. Since AB^{-1} is a diagonalisable matrix of rank 1, $(AB^{-1})^{1/2} = (\operatorname{tr} AB^{-1})^{-1/2} AB^{-1}$. So

$$G(A,B) = (\operatorname{tr} AB^{-1})^{-1/2} AB^{-1} B = (\operatorname{tr} AB^{-1})^{-1/2} A. \qquad \square$$

Lemma 2. *Let $T = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ be a positive semidefinite matrix. Then for all $\alpha \ge 0$ we have*

$$T^\alpha = \frac{1}{2} \begin{bmatrix} (a+b)^\alpha + (a-b)^\alpha & (a+b)^\alpha - (a-b)^\alpha \\ (a+b)^\alpha - (a-b)^\alpha & (a+b)^\alpha + (a-b)^\alpha \end{bmatrix}. \tag{20}$$

Proof. Let U be the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

One can see that $T = UDU^*$ where D is the diagonal matrix

$$D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}. \tag{21}$$

So $T^\alpha = UD^\alpha U^*$, and this is the matrix in (20). □

Example 2. For each $m \geq 0$ and $\varepsilon > 0$ consider the matrices

$$A_m = \begin{bmatrix} 1 & \\ & m \end{bmatrix} \text{ and } B = \begin{bmatrix} 1+\varepsilon/2 & 1-\varepsilon/2 \\ 1-\varepsilon/2 & 1+\varepsilon/2 \end{bmatrix}.$$

Then

$$\lambda_2(A_m \# B) = \frac{1}{\lambda_1(A_m \# B)^{-1}} = \frac{1}{\lambda_1(A_m^{-1} \# B^{-1})}.$$

So, using Lemma 1 we have

$$\lim_{m\to\infty} \lambda_2(A_m \# B) = \frac{1}{\lambda_1(A_0 \# B^{-1})} = \frac{1}{(\mathrm{tr}(A_0 B))^{-1/2} \lambda_1(A_0)} = \left(1 + \frac{\varepsilon}{2}\right)^{1/2} > 1.$$

Let $0 < p < 1$. The inequality above shows that there exists m_0 such that

$$\lambda_2^p(A_{m_0} \# B) > 1.$$

If U is the unitary matrix $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then $U^* B U = \begin{bmatrix} 2 & 0 \\ 0 & \varepsilon \end{bmatrix}$. So $U^* B^p U = \begin{bmatrix} 2^p & 0 \\ 0 & \varepsilon^p \end{bmatrix}$ and

$$B^p = \begin{bmatrix} \dfrac{2^p + \varepsilon^p}{2} & * \\ * & * \end{bmatrix},$$

where the $*$ indicate entries that will not enter the following considerations. We have

$$2\,Q_p^p(A_m, B) = A_m^p + B^p = \begin{bmatrix} 1 + \dfrac{2^p + \varepsilon^p}{2} & * \\ * & * \end{bmatrix}.$$

For any $n \times n$ Hermitian matrix X, we have the inequality $\lambda_n(X) \leq \min_j x_{jj}$. So

$$\lambda_2(2\,Q_p^p(A_m, B)) \leq 1 + \frac{2^p + \varepsilon^p}{2}.$$

Since $0 < p < 1$, we have $2^p < 2$. Choose ε so small that $2^p + \varepsilon^p < 2$. Then we have

$$\lambda_2^p(Q_p(A_m, B)) \leq \frac{1}{2}\left(1 + \frac{2^p + \varepsilon^p}{2}\right) < 1.$$

Thus

$$\lambda_2^p(Q_p(A_{m_0}, B)) < \lambda_2^p(A_{m_0} \# B).$$

We have produced 2×2 matrices A and B for which $\lambda_2(G(A,B)) > \lambda_2(Q_p(A, B))$. This proves Proposition 4.

We will now prove our Proposition 3. We will use the following lemma:

Lemma 3. *Let A be positive semidefinite matrix of rank 1. Let B be any positive definite matrix. Then $F(A,B) = S(A,B)$.*

Proof. If A is of rank 1, then $F(A,B)$ is of rank at most 1. We have

$$\begin{aligned} F(A,B) &= (A^{1/2} B A^{1/2})^{1/2} \\ &= (\operatorname{tr} A^{1/2} B A^{1/2})^{-1/2} A^{1/2} B A^{1/2} \\ &= (\operatorname{tr} AB)^{-1/2} A^{1/2} B A^{1/2}. \end{aligned}$$

By definition

$$S = (A \# B^{-1})^{1/2} B \, (A \# B^{-1})^{1/2},$$

and then by Lemma 1

$$S = (\operatorname{tr} AB)^{-1/4} A^{1/2} B \, (\operatorname{tr} AB)^{-1/4} A^{1/2} = (\operatorname{tr} AB)^{-1/2} A^{1/2} B A^{1/2}.$$

This proves the Lemma. □

Lemma 4. *Let u be a unit vector and let $A = a u u^*$ where $a > 0$. Let B be any positive definite matrix. Let $b = \langle u, Bu \rangle$ and $q = \langle u, Q_p(A,B)\, u \rangle$ where $p > 0$. Then $S(A,B) \preceq Q_p(A,B)$ if and only if $\sqrt{ab} \leq q$.*

Proof. For brevity we will write S and Q_p for $S(A,B)$ and $Q_p(A,B)$. Since A has rank 1, both S and $S \# Q_p^{-1}$ have rank at most 1. Therefore $S \# Q_p^{-1} \leq I$ if and only if

$$\operatorname{tr} S \# Q_p^{-1} \leq 1. \tag{22}$$

To prove the Lemma we have to show that this condition is equivalent to $\sqrt{ab} \le q$.

From Lemma 1 we have

$$\begin{aligned} \operatorname{tr} S\#Q_p^{-1} &= (\operatorname{tr} SQ_p)^{-1/2}(\operatorname{tr} S) \\ &= (\operatorname{tr} SQ_p)^{-1/2}(\operatorname{tr} (AB)^{1/2}) \\ &= (\operatorname{tr} SQ_p)^{-1/2}(\operatorname{tr} (AB))^{1/2}. \end{aligned} \tag{23}$$

(The second inequality in this chain is a consequence of the fact that $S =_\lambda (AB)^{1/2}$, and the third of the fact that AB has rank 1.) Now note that

$$\operatorname{tr} AB = \operatorname{tr} (auu^*)\, B = a \operatorname{tr} u^* Bu = ab. \tag{24}$$

By Lemma 3

$$S = (\operatorname{tr} AB)^{-1/2} A^{1/2} BA^{1/2} = (ab)^{-1/2} A^{1/2} BA^{1/2}.$$

So

$$\begin{aligned} \operatorname{tr} SQ_p &= (ab)^{-1/2} \operatorname{tr} A^{1/2} BA^{1/2} Q_p \\ &= (ab)^{-1/2} \operatorname{tr} a^{1/2} uu^* Ba^{1/2} uu^* Q_p \\ &= (ab)^{-1/2} a \operatorname{tr} u^* Bu \cdot u^* Q_p u \\ &= (ab)^{-1/2} abq \\ &= (ab)^{1/2} q. \end{aligned} \tag{25}$$

From (23), (24) and (25) we see that

$$\operatorname{tr} S\#Q_p^{-1} = (ab)^{-1/4} q^{-1/2} (ab)^{1/2} = (ab)^{1/4} q^{-1/2}.$$

So the condition (22) is equivalent to $\sqrt{ab} \le q$. This proves Lemma 4. □

We will now show that for every $p \in (0,1)$ there exist positive semi-definite matrices $A = auu^*$ and B for which $\sqrt{ab} > q$. Here $b = \langle u, Bu \rangle$ and $q = \langle u, Q_p(A,B)u \rangle$. This is enough to prove Proposition 3. (If X is a metric space with an everywhere dense subset D, and f is a real continuous function on X, then the existence of a point x in X for which $f(x) > 0$ implies the existence of a point y in D for which $f(y) > 0$.)

Example 3. Choose p in $(0,1)$. Let $r = 1/p$. Then $r > 1$. Let A and B be the matrices given by

$$A^p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B^p = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}.$$

Note $P = \frac{1}{3} B^p$ is an orthogonal projection. So

$$B = 3^{\frac{1}{p}} P = 3^{r-1} \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}.$$

Let $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$a := \langle u, Au \rangle = 1 \text{ and } b = \langle u, Bu \rangle = 3^{r-1}.$$

We also have

$$Q_p^p = \frac{A^p + B^p}{2} = \frac{1}{2} \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}.$$

So by Lemma 2

$$Q_p = \frac{1}{2^{1/p}} \begin{bmatrix} \frac{(2+\sqrt{2})^{1/p} + (2-\sqrt{2})^{1/p}}{2} & * \\ * & * \end{bmatrix}.$$

So

$$q = \langle u, Q_p u \rangle = \frac{1}{2^{r+1}} [(2+\sqrt{2})^r + (2-\sqrt{2})^r].$$

If we show $\sqrt{ab} > q$, then Proposition 3 will follow from Lemma 4. That is, we need to show

$$2^{r+1} 3^{(r-1)/2} > (2+\sqrt{2})^r + (2-\sqrt{2})^r$$

for $r > 1$. Dividing both sides by $(2\sqrt{3})^r$, this inequality can be written as

$$\left(\frac{2+\sqrt{2}}{2\sqrt{3}} \right)^r + \left(\frac{2-\sqrt{2}}{2\sqrt{3}} \right)^r < \frac{2}{\sqrt{3}}. \tag{26}$$

Observe that $2+\sqrt{2}$ and $2-\sqrt{2}$ are smaller than $2\sqrt{3}$. So the left hand side is a decreasing function of r on $(0, \infty)$. When $r = 1$, the two sides of (26) are equal. So for all $r > 1$ we have the inequality (26). This establishes our claim and proves Proposition 3.

Finally, we turn to the proof of Proposition 2. If A is a positive semidefinite matrix of rank 1, then Lemma 3 gives $S(A, B) = F(A, B)$. In example 3 we showed that for each $0 < p < 1$ there exist 2×2 matrices A and B of rank 1 for which the inequality $F(A, B) \preceq Q_p(A, B)$ is not true either. We are then left with the case $p \geq 1$.

If a and b are positive numbers, we have

$$Q_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}=\left(\frac{(a/b)^p+1}{2}\right)^{1/p}b\le\max\left(1,\frac{a}{b}\right)b,$$

for all $p\in\mathbb{R}$. The next lemma is an analogue of this for matrices, but we need to restrict to $p\ge 1$.

Lemma 5. *Let A and B be positive definite matrices and let*

$$c=\max\left(1,\frac{\lambda_1(A)}{\lambda_n(B)}\right).$$

Then for all $p\ge 1$, we have

$$Q_p(A,B)\le cB.$$

Proof. We have

$$\lambda_1(A)=\frac{\lambda_1(A)}{\lambda_n(B)}\lambda_n(B)\le c\lambda_n(B)=\lambda_n(cB).$$

This shows that $\|A\|\,I\le cB$, and hence $\|A\|^p I\le(cB)^p$ for all $p>0$. We also have $A^p\le\|A\|^p I$. So $A^p\le(cB)^p$. This gives

$$\frac{A^p+B^p}{2}\le(cB)^p,$$

for all $p>0$. For $p\ge 1$, the function $f(t)=t^{1/p}$ is operator monotone on the positive half-line. Hence the above inequality gives

$$Q_p(A,B)\le cB.$$ □

Example 4. Let $p\ge 1$. For each $k\in\mathbb{N}$ let

$$A_k=\begin{bmatrix}\frac{1}{k^2} & 0\\ 0 & 1\end{bmatrix}\text{ and }B_k=\begin{bmatrix}k^2 & k\\ k & 2\end{bmatrix}.$$

Then

$$F(A_k,B_k)=\left(A_k^{1/2}B_kA_k^{1/2}\right)^{1/2}=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}^{1/2}=\frac{1}{\sqrt{5}}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}.$$

This matrix does not depend on k. Denote it by F for brevity. The inverse of this matrix is

$$F^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

We have $\lambda_1(A_k) = 1$, and a little calculation shows that

$$\lambda_2(B_k) = 1 - \frac{2}{k^2 + \sqrt{k^4+4}} < 1.$$

Thus

$$c_k := \max\left(1, \frac{\lambda_1(A_k)}{\lambda_2(B_k)}\right) = \frac{1}{\lambda_2(B_k)},$$

and $\lim_{k\to\infty} c_k = 1$. By Lemma 5 $Q_p(A_k, B_k) \le c_k B_k$ for all k, and hence

$$F \# Q_p(A_k, B_k)^{-1} \ge F \# (c_k B_k)^{-1}.$$

Hence

$$\|F \# Q_p(A_k, B_k)^{-1}\| \ge \frac{1}{\sqrt{c_k}} \|F \# B_k^{-1}\|. \tag{27}$$

If we show that there exists k for which the right hand side of the inequality (27) is strictly bigger than 1, then it would follow that we do not have $F \# Q_p(A_k, B_k)^{-1} \le I$, and hence we do not have $F \preceq Q_p(A_k, B_k)$. This would prove Proposition 2.

Note that

$$B_k^{-1} = \begin{bmatrix} \frac{2}{k^2} & -\frac{1}{k} \\ -\frac{1}{k} & 1 \end{bmatrix},$$

and so

$$P := \lim_{k\to\infty} B_k^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we have

$$\lim_{k\to\infty} \frac{1}{\sqrt{c_k}} \|F \# B_k^{-1}\| = \|F \# P\|.$$

By Lemma 1 we have

$$\|F \# P\| = \frac{\|P\|}{\sqrt{\operatorname{tr} F^{-1} P}} = \left(\frac{\sqrt{5}}{2}\right)^{1/2}.$$

This quantity is bigger than 1. This proves Proposition 2.

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