Supplementary Material for Real Analysis

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The material in this document will supplement the lectures by the instructor. Although this document will be fairly self-contained, from time to time the knowledge of definitions and concepts covered in the class will be assumed.

Table of contents

1 Defining the real numbers

Question 1.1. What is the difference between an *operation* and a *relation*? Think of examples.

A *group* G is a set equipped with the binary operation $+$ (addition) such that

1. the operation $+$ is associative, that is, for all $a, b, c \in G$, we have $a + (b + c) = (a + b) + c$,
2. there is an (identity) element $0 \in G$ such that $a + 0 = a$ for all $a \in G$,
3. for every $a \in G$, there is an element $-a \in G$ such that $a + (-a) = 0$.

By definition, if G is a group then $|G| \geq 1$. There is no empty group.

A *field* F is a set equipped with binary operations $+$ and $\cdot$ such that

1. $(F, +)$ is an Abelian group (the identity is 0)
2. $(F - \{0\}, \cdot)$ is also an Abelian group (the identity is 1)
3. $\cdot$ distributes over $+$, that is, $a\,(b + c) = ab + ac$ for all $a, b, c \in \mathbb{F}$.

Question 1.2. Why is there no field of cardinality 1?

Again, by definition it is clear that if F is a field then $|F| \geq 2$. A field has at least an additive and a multiplicative identity which are distinct. However, in many other equivalent formulations of the field definition the condition $|F| \geq 2$ is explicitly incorporated. In such formulations, the distinctness of the additive and multiplicative identity is inferred. In fact, if $0 = 1$, then the field necessarily collapses to the singleton $\{0\}$. To see this, note that in F

$$0a = 0 \quad \forall a. \qquad [\textit{Proof. } 0a = (0+0)\,a = 0a + 0a \Longrightarrow 0a = 0]$$

As a corollary, we get that if $0 = 1$, then $F = \{0\}$. In F we also have

$$(-1)\,a = -a \quad \forall a \quad [\textit{Proof. } (-1+1)\,a = 0 \Longrightarrow (-1)\,a = -a],$$

a small result which we will use in the next subsection.

1.1 Ordered fields

An *ordered field* is a field F equipped with a **total** order such that the field operations are compatible with the order in the following manner:

1. if $a \leq b$ then $a + c \leq b + c$ for all c,
2. if $a \leq b$ and $c \geq 0$ then $ac \leq bc$.

So, we have seen that given a set, one can give structure to it in multiple ways. One way is assign operations on the set. In this direction we get groups, fields etc. Another is to give it an ordered, which gives rise to ordered sets. An ordered field combines both the structures. The following diagram displays this as a hierarchy diagram.

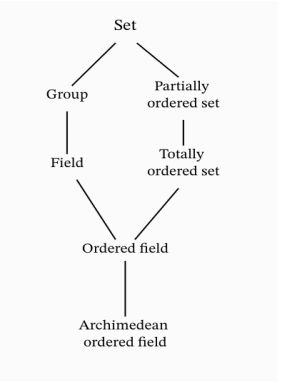

Figure 1.1. Hierarchy diagram of structured sets

We say an ordered field F is *Archimedean* if whenever $x, y \in F$ such that $x < y$, then there exist $n \in \mathbb{N}$ such that $y < nx$. Here, in a general field by nx we mean x added to itself n times.

Informally, the Archimedean property says that no element of the field is either too large or too small. Some properties ordered fields are listed below with their proofs.

$a \leq b \Longrightarrow -a \leq -b$	$a+(-b-a) \leq b+(-b-a) \Longrightarrow -b \leq -a.$
$a \leq b \Longrightarrow bc \leq ac \quad \forall c \leq 0$	$a \leq b \Longrightarrow a(-c) \leq b(-c) \Longrightarrow -ac \leq -bc \Longrightarrow bc \leq ac.$
$a \geq 0,\ b \geq 0 \Longrightarrow ab \geq 0$	$a \geq 0 \Longrightarrow ab \geq 0b \Longrightarrow ab \geq 0.$
$a^2 \geq 0 \quad \forall a$	Only need to show for when $a \leq 0$. In that case $-a \geq 0$. Then $a^2 = (-1)(-1)\,a^2 = (-a)(-a) \geq 0.$
$0 \leq 1$	$(-1)^2 = 1$ Incorrect proof: pick $a \geq 0$ such that $a \neq 0$. Then $0 \geq 1 \Longrightarrow 0 \geq a$. Incorrect because such a choice of a may not exist.
$a \geq 0 \Longrightarrow a^{-1} \geq 0$	If $a^{-1} \leq 0$ then $a^{-1}a \leq 0a \Longrightarrow 1 \leq 0$. Contradiction.
$0 \leq a \leq b \Longrightarrow 0 \leq b^{-1} \leq a^{-1}$	$a, b \geq 0$ then $a^{-1}b^{-1} \geq 0$. Multiply $a \leq b$ with $a^{-1}b^{-1}$.

Note that F being a total order is important in the proof of the statement $a^2 \geq 0$, and (therefore) for all the statements that follow from it.

1.2 Least upper bounds

An element $a \in S \subset W$ is said to be an *upper bound* for S if $a \geq s$ for all $s \in S$. Now, a is called a *least upper bound* or an *infimum* for S if whenever b is another upper bound for S, it holds that $a \leq b$. If S has an infimum it is unique. Therefore we denote this infimum as $\inf S$.

A poset W is said to have the *least upper bound (LUB) property* if whenever a subset S has an upper bound in W, it has a least upper bound in W. The natural analogues of these quantities are *lower bounds, greatest lower bounds or supremums,* and the *greatest lower bound (GLB) property*. The notation for the supremum of the set S is $\sup S$.

The set $\mathbb{Q}$ of rationals does not have the LUB property, while $\mathbb{Z}$ and $\mathbb{N}$ do. In fact, $\mathbb{Z}$ and $\mathbb{N}$ have something stronger. Any non-empty subset S of $\mathbb{Z}$ or $\mathbb{N}$, that has an upper bound in S, has its supremum in S itself. In order to show this we assume the following characterization of the integers.

Proposition 1.1. *$\mathbb{Z}$ is the smallest subring of $\mathbb{R}$ whose positive elements, say $\mathbb{Z}_+$, are well-ordered, that is, if $S \subset \mathbb{Z}_+$ and S is non-empty, then S has a minimum.*

The above also gives a characterisation of the natural numbers as being the positive elements of $\mathbb{Z}$.

Corollary 1.1. *Suppose $s, t \in \mathbb{Z}$. Then, either $s \leq t$ or $s \geq t+1$.*

Proof. Suppose not, that is, suppose $s, t \in \mathbb{Z}$ such that $t < s < t+1$. Consider $r = s - t$. We have $r \in \mathbb{Z}$ and $0 < r < 1$. Let

$$S = \{r, r^2, \dots\}.$$

As $r > 0$, we have $r^n > 0$ for all $n \in \mathbb{N}$. Therefore, $S \subset \mathbb{Z}_+$. By the well-ordering principle, S has a minimum, say r^k. Further, as $r < 1$, we get $r^k r < r^k$ or $r^{k+1} < r^k$. This is a contradiction to the minimality of r^k. □

Proposition 1.2. *Let $S \subset \mathbb{Z}$ be non-empty such that S has an upper bound. Then* $\sup S \in S$.

Proof. We know that $\sup S \in \mathbb{R}$. Call this number α. Note that $\alpha - 1$ is not an upper bound for S, therefore there exist $s \in S$ such that

$$s \leq \alpha \quad \text{and} \quad \alpha - 1 < s.$$

Let $t \in S$. Suppose $t \geq s+1$. Then $\alpha < s+1 \leq t$, which is a contradiction, as α is an upper bound for S. Therefore, we have $t \leq s$. This shows that s is an upper bound for S, which in turn implies that $\alpha \leq s$. Since $\alpha \geq s$ also, we conclude that $\alpha = s$. □

Definition 1.1. The set of real numbers, denoted by $\mathbb{R}$, is the smallest ordered field containing $\mathbb{Q}$ (as an ordered field) that has the LUB property.

Using just this definition alone we can prove many familiar properties of $\mathbb{R}$. For $x \in \mathbb{R}$ let

$$U(x) = \{z \in \mathbb{Z} \mid z \geq x\} \text{ and } L(x) = \{z \in \mathbb{Z} | z \leq x\}.$$

Since $\mathbb{Z}$ has the least upper bound property, we can define the following $\mathbb{Z}$-valued functions on $\mathbb{R}$,

$$\lceil x \rceil = \inf U(x) \text{ and } \lfloor x \rfloor = \sup L(x).$$

These are the familiar *ceiling* and *floor* functions, respectively. Below we list some propeties of these functions.

$\lfloor x \rfloor \leq x \leq \lceil x \rceil$	To see this note that x is a lower bound for $U(x)$. So $$\lceil x \rceil = \inf U(x) \geq x.$$ The other side of the inequality can be similarly argued.
$0 \leq \lceil x \rceil - \lfloor x \rfloor \leq 1$	The left inequality follows from point 1. To see the right one, note that the integer $\lfloor x \rfloor + 1$ is not in $L(x)$, as $\lfloor x \rfloor$ is an upper bound for $L(x)$. Since $\mathbb{Z}$ has a total order, it must be that $\lfloor x \rfloor + 1 \geq x$. Therefore, $\lfloor x \rfloor + 1 \in U(x)$. This shows that $\lceil x \rceil \leq \lfloor x \rfloor + 1$, and $\lceil x \rceil$ is a lower bound for $U(x)$.

Proposition 1.3. *Between any two real numbers there is an irrational number.*

Proof. Suppose $a, b \in \mathbb{R}$ such that $a \leq b$. Let γ be any irrational number. We have $a - \gamma \leq b - \gamma$. We know that there exists $q \in \mathbb{Q}$ such that

$$a - \gamma \leq q \leq b - \gamma.$$

This implies

$$a \leq q + \gamma \leq b.$$

Now, $q + \gamma$ is not a rational number, because if it were, then γ would be a rational number. □

Exercises

Problem 1.1. Suppose U is an Abelian group. Let P be a subset of U with the following properties:

i. $0 \in P$,

ii. if $p \in P$ and $-p \in P$ then $p = 0$,

iii. if $p_1, p_2 \in P$ then $p_1 + p_2 \in P$.

Such a subset is called a *postive cone*. Use P to define a partial order on U.

Conclude (informally) that for an Abelian group, having a partial order is equivalent to having a positive cone.

Problem 1.2. [optional] Given $a, b \in \mathbb{R}$ such that $a < b$, give an algorithm to find a $q \in \mathbb{Q}$ such that $a < q < b$. Argue why your algorithm is correct.

Problem 1.3. Let F be an ordered field. Define the absolute value function $|\cdot|$ on F as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Show the following

1. $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$
2. $|x| = |-x|$
3. $x \leq |x|$
4. $|x + y| \leq |x| + |y|$
5. $|x| \leq a$ if and only if $-a \leq x \leq a$
6. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$.

Give a one line reason why the function $d(x, y) = |x - y|$ is not a valid metric on F.

Problem 1.4. Suppose a non-zero polynomial p is expressed as $p(x) = p_m + p_{m-1}x + \cdots + p_0 x^m$ for some $m \in \mathbb{N}$, where $p_0 \neq 0$. Let

$$F = \left\{ \frac{p}{q} : p, q \text{ polynomials and } q \neq 0 \right\}.$$

1. Show that F is a field.
2. Let $P = \{{}^{p}/_{q} \in F : p_0 q_0 \geq 0\}$. Show that P is a positive cone (refer Exercise II, problem 2). Conclude that F is an ordered field.
3. Show that F does not have the Archimedean property.

Problem 1.5. A theorem due to Hölder states that *every Archimedean ordered field is isomorphic*[1.1] *to a subfield of* $\mathbb{R}$. Use this to conclude that every Archimedean ordered field can be made into a metric space.

1.1. An isomorphism between two ordered fields is a bijection f that preserves the operations and the order, that is, $f(a \cdot b + c) = f(a) \cdot f(b) + f(c)$ and $f(a) \leq f(b)$ whenever $a \leq b$.