Supplementary Material for Fourier Analysis

Pritam Chandra

1 Some rules of interchanging limits

Definition 1. A function $f\colon A \to \mathbb{R}$ is said to be in $C^k(A)$ if f is k times differentiable on A and its kth derivative $f^{(k)}$ is continuous.

Fact 1. *A differentiable function is continuous, but need not be uniformly continuous. An example of such a function is $f(x) = x^2$.*

Fact 2. *Every continuous function on a compact interval is uniformly continuous.*

Rule 1. (Leibniz's rule for differentiating under the integral sign) *Let $f \in C^1(\mathbb{R}^2)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x = \int_a^b \frac{\partial}{\partial t} f(x,t)\,\mathrm{d}x.$$

Rule 2. (Fubini's theorem for interchanging the order of integration) *Let R be the rectangle $[a,b] \times [c,d]$. Let f be a function defined on R. Consider the integrals*

$$\int_R f(x,y)\,\mathrm{d}A(x,y),$$
$$\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$
$$\int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

If any of the following conditions,

(i) f is continuous everywhere on R

(ii) f is continuous everywhere except on a "small" set and f is bounded

(iii) $f \geq 0$ *and one of the three integrals is finite*

(iv) $\int_R |f| \,\mathrm{d}A$ *is finite,*

then all the three integrals equal.

Rule 3. *Let* $f: [a,b] \to \mathbb{R}$ *be continuous. Then for all* $x \in (a,b)$*,*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t) \,\mathrm{d}t = f(x).$$

Rule 4. (Interchanging sums and integrals) *Let* $f_n: \mathbb{R} \to \mathbb{R}$ *be a sequence of functions. Then the equality*

$$\int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x) \,\mathrm{d}x = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x) \,\mathrm{d}x$$

holds if one of the following conditions hold:

(i) $f_n(x) \geq 0$ *for all* $x \in \mathbb{R}$

(ii) the series $\sum \int_{\mathbb{R}} |f_n(x)| < \infty$.

Rule 5. [Lebesgue dominated convergence theorem (LDCT)] *Let* f_n *and* f *be a sequence of measurable functions on the measure space* $(X, \mathcal{B}, \mu)$ *such that* $f_n \to f$ *pointwise. Further, suppose* f_n *is dominated by an integrable function* g*, that is,*

$$|f_n(x)| \leq g(x) \text{ for all } x \in X \text{ and } n \in \mathbb{N}.$$

Then f_n *and* f *are integrable, and*

$$\lim_{n \to \infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu.$$

Rule 6. (Weierstrass M-test) *Let* f_n *be a sequence of real or complex valued functions defined on* A*. Suppose for each* $n \in \mathbb{N}$ *there exists a number* M_n *such that*

$$|f_n(x)| \leq M_n \text{ for all } x \in A,$$

and the series $\sum M_n$ *is convergent. Then the series* $\sum f_n(x)$ *converges absolutely and uniformly on* A*.*

2 Tauber's theorem

Theorem 1. [Tauber] *Suppose the series* $\sum x_n$ *is Abel summable and the sequence* $n x_n$ *converges to 0. Then the series* $\sum x_n$ *is convergent.*

Proof. Let L be the Abel sum of $\sum x_n$. For $0<r<1$, let $L(r)=\sum_{n=1}^{\infty} x_n r^n$. For such an r, note that

$$(1-r^n)=(1-r)(1+r+r^2+\cdots+r^{n-1})\leq n\,(1-r).$$

For any $m\in\mathbb{N}$, let s_m be the partial sums of the series $\sum x_n$ and let $r_m=1-{}^1\!/_m$. We have

$$|s_m-L|\leq|s_m-L(r_m)|+|L-L(r_m)|$$

and

$$|s_m-L(r_m)| \;=\; \left|\sum_{n=1}^{m} x_n-\sum_{n=1}^{\infty} x_n r_m^n\right| \;\leq\; \left|\sum_{n=1}^{m} x_n(1-r_m^n)\right|+\left|\sum_{n=m+1}^{\infty} x_n r_m^n\right|.$$

Pick any $\varepsilon>0$. Firstly, since $n x_n$ converges, it is bounded. That is, there is some B such that $|n x_n|\leq B$ for all n. There also exists N_1 such that $|n x_n|<\frac{\varepsilon}{6}$ for every $n\geq N_1$. Further, there exists $N_2\geq N_1$ such that $\frac{1}{N_2}<\frac{\varepsilon}{6BN_1}$. So, for all $m\geq N_2$

$$\begin{aligned}
\left|\sum_{n=1}^{m} x_n(1-r_m^n)\right| &\leq \sum_{n=1}^{m} |n x_n(1-r_m)|\\
&=\frac{1}{m}\sum_{n=1}^{N_1}|n x_n|+\frac{1}{m}\sum_{n=N_1+1}^{m}|n x_n|\\
&<\frac{N_1 B}{m}+\frac{\varepsilon}{6}\\
&\leq\frac{N_1 B}{N_2}+\frac{\varepsilon}{6}\\
&<\frac{\varepsilon}{3},
\end{aligned}$$

and

$$\left|\sum_{n=m+1}^{\infty} x_n r_m^n\right|\leq\sum_{n=m+1}^{\infty}|x_n|\,r_m^n<\frac{\varepsilon}{6}\sum_{n=m+1}^{\infty}\frac{r_m^n}{n}<\frac{\varepsilon}{6m}\,\frac{r_m^{m+1}}{(1-r_m)}<\frac{\varepsilon}{3}.$$

Here for the second inequality, the bound $|n x_n|<\frac{\varepsilon}{6}$ has been used again noting that $n>m\geq N_1$. Finally, since $L(r_m)$ converges to L, there exists $N_3\geq N_2$ such that for all $m\geq N_3$

$$|L(r_m)-L|<\frac{\varepsilon}{3}.$$

So, for all $m > N_3$ we have

$$\begin{aligned} |s_m - L| &\leq \left|\sum_{n=1}^{m} x_n(1 - r_m^n)\right| + \left|\sum_{n=m+1}^{\infty} x_n r_m^n\right| + |L - L(r_m)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Therefore, L is the sum of series $\sum x_n$. □

3 Cesàro convergence of the Fourier series

The operation $*$, called *convolution*, is a binary operation on $C[-\pi, \pi]$. Suppose f, $g \in C[-\pi, \pi]$. Identify f and g with their extensions to $\mathbb{R}$ with the understanding that $f(\theta) = f(\theta + 2\pi)$ and $g(\theta) = g(\theta + 2\pi)$. Then

$$(f * g)(\theta) = \int_{-\pi}^{\pi} f(t)\, g(\theta - t)\, \mathrm{d}t.$$

The operation $*$ is commutative. There is no identity element corresponding to this operation. Although, there exist families Q_n such that

$$\lim_{n \to \infty} Q_n * f = f.$$

Such a family is called an *approximate identity* for convolution. A special approximate identity is the Dirac family.[1]

For $f \in C[-\pi, \pi]$, define

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, \mathrm{d}t.$$

This is always defined as

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, \mathrm{d}t \right| \leq M,$$

where M is the maximum value taken by the continuous function $|f(t)|$ in the compact interval $[-\pi, \pi]$. Define the series

$$\mathcal{F}f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{int}.$$

1. A Dirac family is a sequence $Q_n \in C[-\pi, \pi]$ that satisfy four conditions: the functions are (i) positive, (ii) even, (iii) normalized (their integral is 1), and (iv) "peaking at 0", that is, for any δ-neighborhood of 0, for large enough n, the integral of Q_n outside the neighborhood is arbitrarily small.

Thus, for every $f \in C[-\pi, \pi]$, the series $\mathcal{F}f$ is defined. This is called the *Fourier series* of f. The numbers $\hat{f}(n)$ are called *Fourier coefficients*.

Question 1. When is it true that $\mathcal{F}f(t) = f(t)$?

On $[-\pi, \pi]$ define the family of functions

$$D_n(t) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikt} \text{ and } F_n(t) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(t).$$

These families are called the *Dirichlet kernel* and the *Fejér kernel* respectively. The Fejér kernel is a Dirac family, whereas the Dirichlet kernel is not (it can take negative values).

Now, for any series $\mathcal{S} = \sum x_n$, let $s_n \mathcal{S}$ be the sequence of partial sums and let $\sigma_n \mathcal{S}$ be the sequence of Cesàro sums. That is,

$$s_n \mathcal{S} = \sum_{k=-n}^{n} x_k \text{ and } \sigma_n \mathcal{S} = \frac{1}{n} \sum_{k=0}^{n-1} s_k \mathcal{S}.$$

It turns out that

$$s_n \mathcal{F}f = D_n * f \text{ and } \sigma_n \mathcal{F}f = F_n * f.$$

Since F_n is a Dirac family, we immediately get Fejér's theorem.

Theorem 2. [Fejér] *For every $f \in C[-\pi, \pi]$, its Fourier series is Cesàro summable to f uniformly. (Alternatively, f is a uniform limit of trigonometric polynomials.)*

Theorem 3. [Weirestrass approximation theorem] *Let $g \in C[-1, 1]$. Then for every $\varepsilon > 0$, there exists a polynomial p such that*

$$\sup_{x \in [-1,1]} |g(x) - p(x)| < \varepsilon.$$

That is, polynomials are dense in the Banach space $C[-1, 1]$ with the supremum norm.

Lemma 1. *For every $n \in \mathbb{N}$, there exists a polynomial $T_n(x)$ such that*

$$\cos nx = T_n(\cos x).$$

Proof. We first note that $T_1(t) = t$ and $T_2(t) = 2t^2 - 1$. Note that

$$\begin{aligned} & \cos(n+1)x + \cos(n-1)x \\ = \; & \cos nx \cos x - \sin nx \sin x + \cos nx \cos x + \sin nx \sin x. \end{aligned}$$

Inductively, let

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t).$$

which is a polynomial satisfying $T_{n+1}(\cos x) = \cos{(n+1)x}$. $\square$

Below we give a proof of Weierstrass approximation theorem using Fejér's theorem.

Proof. Let $g \in C[-1,1]$. Consider $f\colon [-\pi,\pi] \to \mathbb{R}$ such that

$$f(\theta) = g(\cos\theta).$$

Then $f \in C[-\pi,\pi]$. Further, f is even. Recall that

$$\hat{f}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{int}\,\mathrm{d}t.$$

Substitute $u = -t$ and use $f(-u) = f(u)$ to get

$$\hat{f}(-n) = \frac{1}{2\pi}\int_{\pi}^{-\pi} f(-u)\, e^{-inu}\,\mathrm{d}(-u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\, e^{-inu}\mathrm{d}u = \hat{f}(n).$$

Pick any $\varepsilon > 0$. Fejér's theorem tells us that there exists an $N \in \mathbb{N}$ such that

$$\sup_{\theta \in [-\pi,\pi]} |f(\theta) - F_N * f(\theta)| < \frac{\varepsilon}{2}.$$

Using $\hat{f}(n) = \hat{f}(-n)$, we get

$$\begin{aligned}
D_k * f(\theta) &= \sum_{n=-k}^{k} \hat{f}(n)\, e^{in\theta} \\
&= \hat{f}(0) + \sum_{n=1}^{k} \hat{f}(n)(e^{in\theta} + e^{-in\theta}) \\
&= \hat{f}(0) + 2\sum_{n=1}^{k} \hat{f}(n)\cos n\theta.
\end{aligned}$$

But since $\cos n\theta = T_n(\cos\theta)$ for some polynomial T_n, we get that $D_k * f(\theta) = p_k(\cos\theta)$ for some polynomial p_k. So,

$$F_N * f(\theta) = \frac{1}{n}\sum_{k=0}^{n-1} D_k * f(\theta) = \frac{1}{n}(p_0 + \cdots + p_{n-1})(\cos\theta) = p(\cos\theta)$$

for some polynomial p. Now, for any $x \in [-1, 1]$ there exists $\theta \in [-\pi, \pi]$ such that $x = \cos\theta$. So

$$|g(x) - p(x)| = |g(\cos\theta) - p(\cos\theta)| = |f(\theta) - F_N * f(\theta)| < \frac{\varepsilon}{2}.$$

Finally we conclude that

$$\sup_{x \in [-1,1]} |g(x) - p(x)| \leq \frac{\varepsilon}{2} < \varepsilon. \qquad \square$$

4 Problem Set I

Problem 1. Let $\sum x_n$ be a series such that $x_n \geq 0$. Show that if $\sum x_n$ diverges, then it is not Abel summable.

Deduce that for a series with nonnegative terms, the different notions of convergence: usual, Cesàro, and Abel, are all equivalent.

Problem 2. Conclude from problem 1 that if a series is divergent but Abel summable, then it must have infinitely many positive and infinitely many negative terms.

Problem 3. Observe that the conclusion of problem 2 holds if the condition of Abel summability is replaced by Cesàro summability.

Problem 4. Show that if a series is not Cesàro summable but is Abel summable, then it must have infinitely many positive and infinitely many negative terms.

Problem 5. Construct a series $\sum x_n$ such that its partial sums s_N are unbounded (that is $\limsup s_N = \infty$) but its Cesàro sum is 0.

Problem 6. Let $f: [-\pi, \pi] \to \mathbb{R}$ be continuous. Show that $\hat{f}(n) = \overline{\hat{f}(-n)}$. Conclude that the Fourier series of f can be alternatively written as the series

$$\sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Problem 7. Let L_n be the sequence of Lebesgue constants and let D_n be the Dirichlet kernel. Define a new family K_n where

$$K_n(t) = \frac{1}{L_n} |D_n(t)|.$$

Show that K_n is a Dirac family. Ponder on what this means for the convergence of the Fourier series of $f \in C[-\pi, \pi]$.

Problem 8. Recall the centered Gaussian distribution

$$\mathcal{G}_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2\right].$$

Define the kernel G_n as

$$G_n(t) = \mathcal{G}_{\sqrt{1/2n}}(t).$$

Verify G_n is a Dirac family.

Problem 9. (Optional) From complex analysis, a simple consequence of Cauchy estimates is the following. Let $g: \mathbb{C} \to \mathbb{C}$ be an entire[2] function such that

$$|g(x+iy)| \leq \lambda_1 e^{\lambda_2 y^2}$$

for some positive numbers λ_1 and λ_2. Let p_m be the Taylor series of g (around 0) truncated at the mth term. That is,

$$p_m(x) = \sum_{k=0}^{m} \frac{g^{(k)}(0)}{k!} x^k.$$

Then p_m satisfies

$$\sup_{x \in [-\pi,\pi]} |g(x) - p_m(x)| \leq C 2^{-m},$$

for some constant C. Use this and problem 8 to give another proof of Weierstrass approximation theorem. [This approach is close to Weierstrass's original proof].

Problem 10. (Optional) We saw that Fejér's theorem can be used to derive Weirestrass approximation. Derive the same from Poisson's theorem.

2. An entire function $f: \mathbb{C} \to \mathbb{C}$ is a function that is differentiable at every $z \in \mathbb{C}$. This property is stronger than being a C^∞ function as a function of a real variable. However, for the scope of this question, those who are not familiar with complex analysis may assume "entire" to be the same thing as infinitely differentiable.